Sort-Based Confluence Criteria for Non-Left-Linear Higher-Order Rewriting

Thiago Felicissimo & Jean-Pierre Jouannaud

30th International Conference on Automated Deduction
31 July 2025

INRIA

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

The λ-calculus is *confluent*: if $t' \twoheadleftarrow t \twoheadrightarrow t''$ then $t' \twoheadrightarrow t''' \twoheadleftarrow t''$.

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid \mathrm{Eq}(t, u) \mid \top$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid \mathrm{Eq}(t, u) \mid \top$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

Confluence is broken! Known as *Klop's counterexample.*

Let $C := \mathrm{Y}\ (\lambda f x.\mathrm{Eq}(x, f\ x))$ and $A := \mathrm{Y}\ C$. We have:

$$A \twoheadrightarrow C\ A \twoheadrightarrow \mathrm{Eq}(A, C\ A) \twoheadrightarrow \mathrm{Eq}(C\ A, C\ A) \longrightarrow \top$$

$$A \twoheadrightarrow C\ A \twoheadrightarrow \mathrm{Eq}(A, C\ A) \twoheadrightarrow \mathrm{Eq}(\top, C\ \top) \not\longrightarrow \top$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid \top$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid \top$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

Confluence is restored!

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid 0 \mid \mathrm{S}(t)$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid P \vee Q \mid P \wedge Q \mid \top \mid \bot$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

$$\mathrm{Eq}(\mathrm{S}(t), \mathrm{S}(u)) \longrightarrow \mathrm{Eq}(t, u)$$

$$\mathrm{Eq}(\mathrm{S}(t), 0) \longrightarrow \bot$$

$$\mathrm{Eq}(0, \mathrm{S}(t)) \longrightarrow \bot$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid 0 \mid \mathrm{S}(t)$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid P \vee Q \mid P \wedge Q \mid \top \mid \bot$$

$$(\lambda x.t)\ u \longrightarrow t[u/x] \qquad \mathrm{Eq}(\mathrm{S}(t), 0) \longrightarrow \bot$$

$$\mathrm{Eq}(t, t) \longrightarrow \top \qquad \mathrm{Eq}(0, \mathrm{S}(t)) \longrightarrow \bot$$

$$\mathrm{Eq}(\mathrm{S}(t), \mathrm{S}(u)) \longrightarrow \mathrm{Eq}(t, u)$$

Confluence still holds!

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid 0 \mid \mathrm{S}(t)$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid P \vee Q \mid P \wedge Q \mid \top \mid \bot$$

$$(\lambda x.t)\ u \longrightarrow t[u/x]$$

$$\mathrm{Eq}(t, t) \longrightarrow \top$$

$$\mathrm{Eq}(\mathrm{S}(t), \mathrm{S}(u)) \longrightarrow \mathrm{Eq}(t, u)$$

$$\mathrm{Eq}(\mathrm{S}(t), 0) \longrightarrow \bot$$

$$\mathrm{Eq}(0, \mathrm{S}(t)) \longrightarrow \bot$$

$$\mathrm{Eq}(\mathrm{S}(t), t) \longrightarrow \bot$$

A motivating example

$$t, u ::= x \mid t\ u \mid \lambda x.t \mid 0 \mid \mathrm{S}(t)$$

$$P, Q ::= \alpha \mid \mathrm{Eq}(t, u) \mid P \vee Q \mid P \wedge Q \mid \top \mid \bot$$

$$(\lambda x.t)\ u \longrightarrow t[u/x] \qquad \mathrm{Eq}(\mathrm{S}(t), 0) \longrightarrow \bot$$

$$\mathrm{Eq}(t, t) \longrightarrow \top \qquad \mathrm{Eq}(0, \mathrm{S}(t)) \longrightarrow \bot$$

$$\mathrm{Eq}(\mathrm{S}(t), \mathrm{S}(u)) \longrightarrow \mathrm{Eq}(t, u) \qquad \mathrm{Eq}(\mathrm{S}(t), t) \longrightarrow \bot$$

Confluence is broken! Similar to *Huet's counterexample*.

Define $\infty := \mathrm{Y}\ (\lambda x.\mathrm{S}(x))$.

$$\bot \longleftarrow \mathrm{Eq}(\mathrm{S}(\infty), \infty) \longrightarrow\!\!\!\!\to \mathrm{Eq}(\mathrm{S}(\infty), \mathrm{S}(\infty)) \longrightarrow \top$$

This work

I hope I have convinced you that:

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

[1] In the absence of termination/strong normalization.

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

Our contribution Criteria for non-left-linearity that exploit separation into sorts.

[1] In the absence of termination/strong normalization.

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

Our contribution Criteria for non-left-linearity that exploit separation into sorts.

Given $\mathcal{R}$, let $\mathcal{R}_c$ be rules that can apply inside non-linear metavariables, and $\mathcal{R}_{nc}$ other ones.

[1] In the absence of termination/strong normalization.

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

Our contribution Criteria for non-left-linearity that exploit separation into sorts.

Given $\mathcal{R}$, let $\mathcal{R}_c$ be rules that can apply inside non-linear metavariables, and $\mathcal{R}_{nc}$ other ones.

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.

[1] In the absence of termination/strong normalization.

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

Our contribution Criteria for non-left-linearity that exploit separation into sorts.

Given $\mathcal{R}$, let $\mathcal{R}_c$ be rules that can apply inside non-linear metavariables, and $\mathcal{R}_{nc}$ other ones.

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.

[1] In the absence of termination/strong normalization.

This work

I hope I have convinced you that:

Non-left-linear higher-order confluence is subtle![1]

Our contribution Criteria for non-left-linearity that exploit separation into sorts.

Given $\mathcal{R}$, let $\mathcal{R}_c$ be rules that can apply inside non-linear metavariables, and $\mathcal{R}_{nc}$ other ones.

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

[1] In the absence of termination/strong normalization.

Talk overview

The rewriting formalism

The first criterion

The second criterion

The combination criterion

Conclusion

The rewriting formalism

Second-Order Multi-Sorted Terms

Terms $t \in \mathcal{T}_s$ of sort s defined by:

Second-Order Multi-Sorted Terms

Terms $t \in \mathcal{T}_s$ of sort s defined by:

$$\frac{x : s}{x \in \mathcal{T}_s}$$

Second-Order Multi-Sorted Terms

Terms $t \in \mathcal{T}_s$ of sort s defined by:

$$\frac{x : s}{x \in \mathcal{T}_s} \qquad \frac{\mathsf{X} : \{s_1, \ldots, s_k\}s \qquad t_i \in \mathcal{T}_{s_i} \quad (i = 1, \ldots, k)}{\mathsf{X}\{t_1, \ldots, t_k\} \in \mathcal{T}_s}$$

Second-Order Multi-Sorted Terms

Terms $t \in \mathcal{T}_s$ of sort s defined by:

$$\frac{x : s}{x \in \mathcal{T}_s} \qquad \frac{\begin{array}{c} \mathsf{X} : \{s_1, \dots, s_k\}s \\ t_i \in \mathcal{T}_{s_i} \quad (i = 1, \dots, k) \end{array}}{\mathsf{X}\{t_1, \dots, t_k\} \in \mathcal{T}_s} \qquad \frac{\begin{array}{c} f : (\{\vec{s}_1\}s'_1, \dots, \{\vec{s}_k\}s'_k)s \in \Sigma^{2} \\ \vec{x}_i : \vec{s}_i \quad t_i \in \mathcal{T}_{s'_i} \quad (i = 1, \dots, k) \end{array}}{f(\vec{x}_1.t_1, \dots, \vec{x}_k.t_k) \in \mathcal{T}_s}$$

[2] Supposing user-defined (binding/second-order) signature.

Second-Order Multi-Sorted Terms

Terms $t \in \mathcal{T}_s$ of sort s defined by:

$$\frac{x : s}{x \in \mathcal{T}_s} \qquad \frac{\mathsf{X} : \{s_1, \ldots, s_k\}s \qquad t_i \in \mathcal{T}_{s_i} \quad (i = 1, \ldots, k)}{\mathsf{X}\{t_1, \ldots, t_k\} \in \mathcal{T}_s} \qquad \frac{f : (\{\vec{s}_1\}s'_1, \ldots, \{\vec{s}_k\}s'_k)s \in \Sigma^{2} \qquad \vec{x}_i : \vec{s}_i \quad t_i \in \mathcal{T}_{s'_i} \quad (i = 1, \ldots, k)}{f(\vec{x}_1.t_1, \ldots, \vec{x}_k.t_k) \in \mathcal{T}_s}$$

Example

$\Sigma := \{@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm},\ \lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm},\ \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop},\ \wedge : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}\}$

Meta-free terms (with no metavariables) can be described as:

$$\begin{aligned} \mathcal{T}_{\mathrm{tm}} \ni t, u &::= x \in \mathcal{V}_{\mathrm{tm}} \mid @(t, u) \mid \lambda(x.t) \\ \mathcal{T}_{\mathrm{prop}} \ni P, Q &::= \alpha \in \mathcal{V}_{\mathrm{prop}} \mid \mathrm{Eq}(t, u) \mid \wedge(P, Q) \end{aligned}$$

[2] Supposing user-defined (binding/second-order) signature.

Rewrite systems

A *rewrite rule* is a pair $\ell \longrightarrow r$ such that:

- ℓ is a (fully-extended) pattern headed by a function symbol
- $\mathrm{mv}(r) \subseteq \mathrm{mv}(\ell)$ and $\mathrm{fv}(r) = \mathrm{fv}(\ell) = \emptyset$
- ℓ and r are of the same sort

A *rewrite system* $\mathcal{R}$ is a set of rewrite rules.

Rewrite systems

A *rewrite rule* is a pair $\ell \longrightarrow r$ such that:

- ℓ is a (fully-extended) pattern headed by a function symbol
- $\mathrm{mv}(r) \subseteq \mathrm{mv}(\ell)$ and $\mathrm{fv}(r) = \mathrm{fv}(\ell) = \emptyset$
- ℓ and r are of the same sort

A *rewrite system* $\mathcal{R}$ is a set of rewrite rules.

Example

Considering Σ shown previously, we can take

$$\mathcal{R} := \quad @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\} \qquad\qquad \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$$

Rewrite systems

A *rewrite rule* is a pair $\ell \longrightarrow r$ such that:

- ℓ is a (fully-extended) pattern headed by a function symbol
- $\mathrm{mv}(r) \subseteq \mathrm{mv}(\ell)$ and $\mathrm{fv}(r) = \mathrm{fv}(\ell) = \emptyset$
- ℓ and r are of the same sort

A *rewrite system* $\mathcal{R}$ is a set of rewrite rules.

Example

Considering Σ shown previously, we can take

$$\mathcal{R} := \quad @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\} \qquad\qquad \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$$

$\mathcal{R}$ is *confluent* when $\underset{\mathcal{R}}{\longrightarrow}$, the relation generated by $\mathcal{R}$, is confluent over meta-free terms.

The first criterion

Critical pairs

Disclaimer I will be sloppy with bound variables, see the paper for the real definitions.

Critical pairs

Disclaimer I will be sloppy with bound variables, see the paper for the real definitions.

Given patterns t_1 and t_2, we say that t_1 *overlaps* t_2 *at* p when

- p is a functional position of t_2;
- the unification problem $t_1 \overset{?}{=} t_2|_p$ admits a unifier.

Critical pairs

Disclaimer I will be sloppy with bound variables, see the paper for the real definitions.

Given patterns t_1 and t_2, we say that t_1 *overlaps* t_2 *at* p when

- p is a functional position of t_2;
- the unification problem $t_1 \overset{?}{=} t_2|_p$ admits a unifier.

Assume rules $\ell_1 \longrightarrow r_1, \ell_2 \longrightarrow r_2$ and p a position such that ℓ_1 overlaps ℓ_2 at p and, if $p = \varepsilon$, then the rules are not the same. Letting θ be the mgu of $\ell_1 \overset{?}{=} \ell_2|_p$, we have the *critical pair*

$$\langle \ell_2 \{\!| r_1 |\!\}_p [\theta], r_2[\theta] \rangle$$

Critical pairs

Disclaimer I will be sloppy with bound variables, see the paper for the real definitions.

Given patterns t_1 and t_2, we say that t_1 *overlaps* t_2 *at* p when

- p is a functional position of t_2;
- the unification problem $t_1 \overset{?}{=} t_2|_p$ admits a unifier.

Assume rules $\ell_1 \longrightarrow r_1, \ell_2 \longrightarrow r_2$ and p a position such that ℓ_1 overlaps ℓ_2 at p and, if $p = \varepsilon$, then the rules are not the same. Letting θ be the mgu of $\ell_1 \overset{?}{=} \ell_2|_p$, we have the *critical pair*

$$\langle \ell_2 \{\!\{ r_1 \}\!\}_p [\theta], r_2[\theta] \rangle$$

Example

The lhs of $g(\mathsf{Y}, \mathsf{Y}) \longrightarrow \mathsf{Y}$ overlaps the lhs of $f(g(\mathsf{X}, b)) \longrightarrow a$ at position 1.

Indeed, the problem $g(\mathsf{Y}, \mathsf{Y}) \overset{?}{=} g(\mathsf{X}, b)$ admits the unifier $\theta = \{\_ \mapsto b\}$.

This is actually a mgu, so we get the critical pair $\langle f(b), a \rangle$.

Sort confinement

We define the sort pre-order $\preceq$ by

$$s \preceq s' \qquad \iff \qquad \text{for some meta-free } t \in \mathcal{T}_{s'} \text{ we have } t|_p \in \mathcal{T}_s$$

Sort confinement

We define the sort pre-order $\preceq$ by

$$s \preceq s' \qquad \Longleftrightarrow \qquad \text{for some meta-free } t \in \mathcal{T}_{s'} \text{ we have } t|_p \in \mathcal{T}_s$$

We define the set of *confined sorts* $\mathcal{S}_c(\mathcal{R})$ by $s \in \mathcal{S}_c(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \to r \in \mathcal{R}$:

- Some metavariable of sort s' occurs non-linearly in ℓ, or
- Some metavariable of sort s' occurs non-linearly or nested in r.

Sort confinement

We define the sort pre-order $\preceq$ by

$$s \preceq s' \qquad \Longleftrightarrow \qquad \text{for some meta-free } t \in \mathcal{T}_{s'} \text{ we have } t|_p \in \mathcal{T}_s$$

We define the set of *confined sorts* $\mathcal{S}_c(\mathcal{R})$ by $s \in \mathcal{S}_c(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \to r \in \mathcal{R}$:

- Some metavariable of sort s' occurs non-linearly in ℓ, or
- Some metavariable of sort s' occurs non-linearly or nested in r.

Example

$$\begin{array}{lll} \top : \mathrm{prop} & @ : (\mathrm{tm},\mathrm{tm})\mathrm{tm} & \mathrm{Eq}(\mathsf{N},\mathsf{N}) \longrightarrow \top \\ \mathrm{Eq} : (\mathrm{tm},\mathrm{tm})\mathrm{prop} & \lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm} & @(\lambda(x.\mathsf{T}\{x\}),\mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\} \end{array}$$

We have $\mathcal{S}_c(\mathcal{R}) = \{\mathrm{tm}\}$ and $\mathcal{R}_c = \{\mathrm{Eq}(\mathsf{N},\mathsf{N}) \longrightarrow \top\}$.

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent and terminating.*

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent and terminating.*
2. *For all $\ell \longrightarrow r \in \mathcal{R}_{nc}$ and $\ell_c \longrightarrow r_c \in \mathcal{R}_c$, ℓ_c does not overlap ℓ.*

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent and terminating.*
2. *For all $\ell \to r \in \mathcal{R}_{nc}$ and $\ell_c \to r_c \in \mathcal{R}_c$, ℓ_c does not overlap ℓ.*
3. *If $\langle u_1, u_2 \rangle$ is a critical pair of $\ell_1 \to_{\alpha_1} r_1, \ell_2 \to_{\alpha_2} r_2 \in \mathcal{R}_{nc}$, then*

$$u_1 \underset{<\alpha_1}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_1' \xrightarrow[\alpha_2]{q_1}{}^{=} u_1'' \underset{<\alpha_1,\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2'' \;{}^{=}\!\xleftarrow[\alpha_1]{q_2} u_2' \underset{<\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2$$

for some u_1', u_1'', u_2', u_2'' with q_i non-nested in u_i'.

The first criterion

A *labelling* for $\mathcal{R}$ is an assignment from rules in $\mathcal{R}_{nc}$ into a well-founded set $\mathcal{L}$ of *labels*. Rules in $\mathcal{R}_c$ are treated as smaller than the ones in $\mathcal{R}_{nc}$.

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent and terminating.*
2. *For all $\ell \longrightarrow r \in \mathcal{R}_{nc}$ and $\ell_c \longrightarrow r_c \in \mathcal{R}_c$, ℓ_c does not overlap ℓ.*
3. *If $\langle u_1, u_2 \rangle$ is a critical pair of $\ell_1 \longrightarrow_{\alpha_1} r_1, \ell_2 \longrightarrow_{\alpha_2} r_2 \in \mathcal{R}_{nc}$, then*

$$u_1 \underset{<\alpha_1}{\twoheadleftarrow\!\!\twoheadrightarrow} u_1' \xrightarrow[\alpha_2]{q_1}{}^{=} u_1'' \underset{<\alpha_1,\alpha_2}{\twoheadleftarrow\!\!\twoheadrightarrow} u_2'' \;{}^{=}\!\xleftarrow[\alpha_1]{q_2} u_2' \underset{<\alpha_2}{\twoheadleftarrow\!\!\twoheadrightarrow} u_2$$

for some u_1', u_1'', u_2', u_2'' with q_i non-nested in u_i'.

Then $\mathcal{R}$ is confluent.

An example

Example

$$\begin{array}{llll}
0 : \mathrm{tm} & \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathrm{S}(\mathtt{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathrm{S}(\mathtt{N}), 0) \longrightarrow \bot \\
\mathrm{S} : (\mathrm{tm})\mathrm{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathtt{N}, \mathtt{M}) \longrightarrow \mathrm{Eq}(\mathtt{M}, \mathtt{N}) & 4 : \mathrm{Eq}(\mathtt{N}, \mathtt{N}) \longrightarrow \top
\end{array}$$

An example

Example

$$\begin{array}{llll}
0 : \mathrm{tm} & \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathrm{S}(\mathtt{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathrm{S}(\mathtt{N}), 0) \longrightarrow \bot \\
\mathrm{S} : (\mathrm{tm})\mathrm{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathtt{N}, \mathtt{M}) \longrightarrow \mathrm{Eq}(\mathtt{M}, \mathtt{N}) & 4 : \mathrm{Eq}(\mathtt{N}, \mathtt{N}) \longrightarrow \top
\end{array}$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

An example

Example

$$\begin{array}{llll} 0 : \mathrm{tm} & \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathsf{S}(\mathtt{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathsf{S}(\mathtt{N}), 0) \longrightarrow \bot \\ \mathsf{S} : (\mathrm{tm})\mathrm{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathtt{N}, \mathtt{M}) \longrightarrow \mathrm{Eq}(\mathtt{M}, \mathtt{N}) & 4 : \mathrm{Eq}(\mathtt{N}, \mathtt{N}) \longrightarrow \top \end{array}$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

Three critical pairs (ignoring symmetric counterparts):

An example

Example

$$\begin{array}{llll} 0 : \mathrm{tm} & \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathrm{S}(\mathtt{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathrm{S}(\mathtt{N}), 0) \longrightarrow \bot \\ \mathrm{S} : (\mathrm{tm})\mathrm{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathtt{N}, \mathtt{M}) \longrightarrow \mathrm{Eq}(\mathtt{M}, \mathtt{N}) & 4 : \mathrm{Eq}(\mathtt{N}, \mathtt{N}) \longrightarrow \top \end{array}$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

Three critical pairs (ignoring symmetric counterparts):

- $\langle \mathrm{Eq}(0, \mathrm{S}(\mathtt{X})), \bot \rangle$ with rules 2 and 3, closes as $\mathrm{Eq}(0, \mathrm{S}(\mathtt{X})) \xrightarrow{\varepsilon} \bot$ with rule 1.

An example

Example

$$\begin{array}{llll}
0 : \mathrm{tm} & \mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathrm{S}(\mathsf{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathrm{S}(\mathsf{N}), 0) \longrightarrow \bot \\
\mathrm{S} : (\mathrm{tm})\mathrm{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathsf{N}, \mathsf{M}) \longrightarrow \mathrm{Eq}(\mathsf{M}, \mathsf{N}) & 4 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top
\end{array}$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

Three critical pairs (ignoring symmetric counterparts):

- $\langle \mathrm{Eq}(0, \mathrm{S}(\mathsf{X})), \bot \rangle$ with rules 2 and 3, closes as $\mathrm{Eq}(0, \mathrm{S}(\mathsf{X})) \xrightarrow{\varepsilon} \bot$ with rule 1.
- $\langle \mathrm{Eq}(\mathrm{S}(\mathsf{X}), 0), \bot \rangle$ with rules 2 and 1, closes as $\mathrm{Eq}(\mathrm{S}(\mathsf{X}), 0) \xrightarrow{\varepsilon} \bot$ with rule 3.

An example

Example

$$\begin{array}{llll}
0 : \mathsf{tm} & \mathrm{Eq} : (\mathsf{tm}, \mathsf{tm})\mathrm{prop} & 1 : \mathrm{Eq}(0, \mathrm{S}(\mathsf{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathrm{S}(\mathsf{N}), 0) \longrightarrow \bot \\
\mathrm{S} : (\mathsf{tm})\mathsf{tm} & \top, \bot : \mathrm{prop} & 2 : \mathrm{Eq}(\mathsf{N}, \mathsf{M}) \longrightarrow \mathrm{Eq}(\mathsf{M}, \mathsf{N}) & 4 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top
\end{array}$$

We have $\mathcal{S}_c = \{\mathsf{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

Three critical pairs (ignoring symmetric counterparts):

- $\langle \mathrm{Eq}(0, \mathrm{S}(\mathsf{X})), \bot \rangle$ with rules 2 and 3, closes as $\mathrm{Eq}(0, \mathrm{S}(\mathsf{X})) \xrightarrow{\varepsilon} \bot$ with rule 1.
- $\langle \mathrm{Eq}(\mathrm{S}(\mathsf{X}), 0), \bot \rangle$ with rules 2 and 1, closes as $\mathrm{Eq}(\mathrm{S}(\mathsf{X}), 0) \xrightarrow{\varepsilon} \bot$ with rule 3.
- $\langle \mathrm{Eq}(\mathsf{X}, \mathsf{X}), \top \rangle$ with rules 2 and 4, closes as $\mathrm{Eq}(\mathsf{X}, \mathsf{X}) \xrightarrow{\varepsilon} \top$ with rule 4.

An example

Example

$$\begin{array}{llll} 0 : \mathsf{tm} & \mathrm{Eq} : (\mathsf{tm}, \mathsf{tm})\mathsf{prop} & 1 : \mathrm{Eq}(0, \mathsf{S}(\mathsf{N})) \longrightarrow \bot & 3 : \mathrm{Eq}(\mathsf{S}(\mathsf{N}), 0) \longrightarrow \bot \\ \mathsf{S} : (\mathsf{tm})\mathsf{tm} & \top, \bot : \mathsf{prop} & 2 : \mathrm{Eq}(\mathsf{N}, \mathsf{M}) \longrightarrow \mathrm{Eq}(\mathsf{M}, \mathsf{N}) & 4 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top \end{array}$$

We have $\mathcal{S}_c = \{\mathsf{tm}\}$ hence $\mathcal{R}_c$ empty, so we only need to verify the last condition.

Three critical pairs (ignoring symmetric counterparts):

- $\langle \mathrm{Eq}(0, \mathsf{S}(\mathsf{X})), \bot \rangle$ with rules 2 and 3, closes as $\mathrm{Eq}(0, \mathsf{S}(\mathsf{X})) \xrightarrow{\varepsilon} \bot$ with rule 1.
- $\langle \mathrm{Eq}(\mathsf{S}(\mathsf{X}), 0), \bot \rangle$ with rules 2 and 1, closes as $\mathrm{Eq}(\mathsf{S}(\mathsf{X}), 0) \xrightarrow{\varepsilon} \bot$ with rule 3.
- $\langle \mathrm{Eq}(\mathsf{X}, \mathsf{X}), \top \rangle$ with rules 2 and 4, closes as $\mathrm{Eq}(\mathsf{X}, \mathsf{X}) \xrightarrow{\varepsilon} \top$ with rule 4.

We conclude confluence by labeling rules 1 and 3 the same.

The second criterion

Cyclic unification

Given term t, write $\hat{t}$ for its *linearization*, obtained by replacing X in position p by X_p.

Cyclic unification

Given term t, write $\hat{t}$ for its *linearization*, obtained by replacing X in position p by X_p.

Given disjoint patterns t_1, t_2 and rewrite system $\mathcal{R}$,
an $\mathcal{R}$-*cyclic unifier* of $t_1 \overset{?}{=} t_2$ is a syntactic unifier θ of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ st:

$$\text{for all } \mathsf{X}_p, \mathsf{X}_q \in \mathsf{mv}(\hat{t}_1) \cup \mathsf{mv}(\hat{t}_2) : \quad \theta(\mathsf{X}_p) \underset{\mathcal{R}}{\twoheadleftarrow\!\!\twoheadrightarrow} \theta(\mathsf{X}_q)$$

Cyclic unification

Given term t, write $\hat{t}$ for its *linearization*, obtained by replacing X in position p by X_p.

Given disjoint patterns t_1, t_2 and rewrite system $\mathcal{R}$,
an $\mathcal{R}$-*cyclic unifier* of $t_1 \overset{?}{=} t_2$ is a syntactic unifier θ of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ st:

$$\text{for all } \mathsf{X}_p, \mathsf{X}_q \in \mathsf{mv}(\hat{t}_1) \cup \mathsf{mv}(\hat{t}_2) : \quad \theta(\mathsf{X}_p) \underset{\mathcal{R}}{\twoheadleftarrow\!\!\twoheadrightarrow} \theta(\mathsf{X}_q)$$

Example

Linearizing $\mathrm{Eq}(\mathsf{X}, \mathsf{X}) \overset{?}{=} \mathrm{Eq}(\mathsf{Y}, \mathrm{S}(\mathsf{Y}))$ yields $\mathrm{Eq}(\mathsf{X}_1, \mathsf{X}_2) \overset{?}{=} \mathrm{Eq}(\mathsf{Y}_1, \mathrm{S}(\mathsf{Y}_{2\cdot 1}))$.

Supposing $\mathrm{S}(\infty) \underset{\mathcal{R}}{\twoheadleftarrow\!\!\twoheadrightarrow} \infty$, then $\theta := \{\mathsf{X}_2 \mapsto \mathrm{S}(\infty), \_ \mapsto \infty\}$ is an $\mathcal{R}$-cyclic unifier.

Description of cyclic unifiers (dcu)

Given a cyclic unification problem $t_1 \overset{?}{=} t_2$, let θ be a (syntactic) mgu of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ (if it exists).

Description of cyclic unifiers (dcu)

Given a cyclic unification problem $t_1 \overset{?}{=} t_2$, let θ be a (syntactic) mgu of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ (if it exists).

Let $\simeq$ be the congruence obtained by simplifying

$$\{\theta(\mathsf{X}_p) \approx \theta(\mathsf{X}_q) \mid \mathsf{X}_p, \mathsf{X}_q \in \mathsf{mv}(\hat{t}_1) \cup \mathsf{mv}(\hat{t}_2)\}$$

with the rule

$$\{f(\vec{x}_1.t_1, \ldots, \vec{x}_k.t_k) \approx f(\vec{x}_1.t'_1, \ldots, \vec{x}_k.t'_k)\} \cup C \quad \rightsquigarrow \quad \{t_i \approx t'_i\}_{i=1,\ldots,k} \cup C$$

Description of cyclic unifiers (dcu)

Given a cyclic unification problem $t_1 \overset{?}{=} t_2$, let θ be a (syntactic) mgu of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ (if it exists).

Let $\simeq$ be the congruence obtained by simplifying

$$\{\theta(\mathsf{X}_p) \approx \theta(\mathsf{X}_q) \mid \mathsf{X}_p, \mathsf{X}_q \in \mathsf{mv}(\hat{t}_1) \cup \mathsf{mv}(\hat{t}_2)\}$$

with the rule

$$\{f(\vec{x}_1.t_1, \ldots, \vec{x}_k.t_k) \approx f(\vec{x}_1.t'_1, \ldots, \vec{x}_k.t'_k)\} \cup C \quad \rightsquigarrow \quad \{t_i \approx t'_i\}_{i=1,\ldots,k} \cup C$$

We call $(\theta, \simeq)$ the *description of cyclic unifiers (dcu)* of $t_1 \overset{?}{=} t_2$.

Description of cyclic unifiers (dcu)

Given a cyclic unification problem $t_1 \overset{?}{=} t_2$, let θ be a (syntactic) mgu of $\hat{t}_1 \overset{?}{=} \hat{t}_2$ (if it exists).

Let $\simeq$ be the congruence obtained by simplifying

$$\{\theta(\mathsf{X}_p) \approx \theta(\mathsf{X}_q) \mid \mathsf{X}_p, \mathsf{X}_q \in \mathsf{mv}(\hat{t}_1) \cup \mathsf{mv}(\hat{t}_2)\}$$

with the rule

$$\{f(\vec{x}_1.t_1, \ldots, \vec{x}_k.t_k) \approx f(\vec{x}_1.t'_1, \ldots, \vec{x}_k.t'_k)\} \cup C \quad \rightsquigarrow \quad \{t_i \approx t'_i\}_{i=1,\ldots,k} \cup C$$

We call $(\theta, \simeq)$ the *description of cyclic unifiers (dcu)* of $t_1 \overset{?}{=} t_2$.

Lemma

Suppose $\mathcal{R}$ confluent and t_1, t_2 patterns st $\hat{t}_1, \hat{t}_2$ is not overlapped by $\mathcal{R}$.

- *If $t_1 \overset{?}{=} t_2$ admits a cyclic unifier, then it admits a dcu $(\theta, \simeq)$.*
- *A meta-free σ is a cyclic unifier of $t_1 \overset{?}{=} t_2$ iff*
  there is a meta-free τ that satisfies $\sigma = \theta[\tau]$ and st $u \simeq v$ implies $u[\tau] \underset{\mathcal{R}}{\twoheadleftarrow\!\!\twoheadrightarrow} v[\tau]$.

Cyclic critical pairs

Given patterns t_1, t_2, we say that t_1 *$\mathcal{R}$-cyclicly overlaps t_2 at p* if

- p is a functional position of t_2;
- the cyclic unification problem $t_1 \overset{?}{=} t_2|_p$ admits some $\mathcal{R}$-cyclic unifier.

Cyclic critical pairs

Given patterns t_1, t_2, we say that *t_1 $\mathcal{R}$-cyclicly overlaps t_2 at p* if

- p is a functional position of t_2;
- the cyclic unification problem $t_1 \overset{?}{=} t_2|_p$ admits some $\mathcal{R}$-cyclic unifier.

Assume rules $\ell_1 \longrightarrow r_1, \ell_2 \longrightarrow r_2$ and p a position such that ℓ_1 $\mathcal{R}$-cyclicly overlaps ℓ_2 at p and, if $p = \varepsilon$, then the rules are not the same. We then have the *$\mathcal{R}$-cyclic critical pair*

$$\langle \ell_2 \{\!| r_1 |\!\}_p [\theta^\#], r_2[\theta^\#], \simeq \rangle$$

with $(\theta, \simeq)$ a dcu for $\ell_1 \overset{?}{=} \ell_2|_p$, and $\theta^\#$ the truncation of θ from $\mathrm{mv}(\hat{\ell}_1 \overset{?}{=} \hat{\ell}_2)$ to $\mathrm{mv}(\ell_1 \overset{?}{=} \ell_2)$.

Cyclic critical pairs

Given patterns t_1, t_2, we say that t_1 *$\mathcal{R}$-cyclicly overlaps* t_2 *at* p if

- p is a functional position of t_2;
- the cyclic unification problem $t_1 \overset{?}{=} t_2|_p$ admits some $\mathcal{R}$-cyclic unifier.

Assume rules $\ell_1 \longrightarrow r_1, \ell_2 \longrightarrow r_2$ and p a position such that ℓ_1 $\mathcal{R}$-cyclicly overlaps ℓ_2 at p and, if $p = \varepsilon$, then the rules are not the same. We then have the *$\mathcal{R}$-cyclic critical pair*

$$\langle \ell_2\{\!\!|\, r_1 \,|\!\!\}_p[\theta^\#], r_2[\theta^\#], \simeq \rangle$$

with $(\theta, \simeq)$ a dcu for $\ell_1 \overset{?}{=} \ell_2|_p$, and $\theta^\#$ the truncation of θ from $\mathrm{mv}(\hat{\ell}_1 \overset{?}{=} \hat{\ell}_2)$ to $\mathrm{mv}(\ell_1 \overset{?}{=} \ell_2)$.

Example

The lhs $\mathsf{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{S}(\mathsf{Z})) \longrightarrow \mathsf{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{Z})$ cyclicly overlaps the lhs of $\mathsf{Eq}_3(\mathsf{Y}, \mathsf{Y}, \mathsf{Y}) \longrightarrow \top$, supposing ∞ with $\mathsf{S}(\infty) \twoheadleftarrow\!\!\!\twoheadrightarrow \infty$.

We have a dcu $(\theta, \simeq)$ with $\theta^\# = \{\mathsf{X} \mapsto \mathsf{W}_1, \mathsf{Z} \mapsto \mathsf{W}_3, \mathsf{Y} \mapsto \mathsf{S}(\mathsf{W}_1)\}$ and $\mathsf{W}_1 \simeq \mathsf{W}_3 \simeq \mathsf{S}(\mathsf{W}_3)$.

Therefore, we have the cyclic critical pair $\langle \mathsf{Eq}_3(\mathsf{S}(\mathsf{W}_1), \mathsf{W}_1, \mathsf{W}_3), \top, \simeq \rangle$.

The criterion

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

The criterion

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. $\mathcal{R}_c$ *is confluent.*

The criterion

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent.*
2. *For all $\ell \longrightarrow r \in \mathcal{R}_{nc}$ and $\ell_c \longrightarrow r_c \in \mathcal{R}_c$, ℓ_c does not overlap $\hat{\ell}$.*

The criterion

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent.*
2. *For all $\ell \longrightarrow r \in \mathcal{R}_{nc}$ and $\ell_c \longrightarrow r_c \in \mathcal{R}_c$, ℓ_c does not overlap $\hat{\ell}$.*
3. *If $\langle u_1, u_2, \simeq \rangle$ is an $\mathcal{R}_c$-cyclic critical pair of $\ell_1 \longrightarrow_{\alpha_1} r_1, \ell_2 \longrightarrow_{\alpha_2} r_2 \in \mathcal{R}_{nc}$, then*

$$u_1 \underset{\simeq;<\alpha_1}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_1' \xrightarrow[\alpha_2]{q_1}{}^{=} u_1'' \underset{\simeq;<\alpha_1,\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2'' \;{}^{=}\!\xleftarrow[\alpha_1]{q_1} u_2' \underset{\simeq;<\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2$$

for some u_1', u_1'', u_2', u_2'' with q_i non-nested in u_i'.

The criterion

Theorem

Given some labeled rewrite system $\mathcal{R}$, suppose that:

1. *$\mathcal{R}_c$ is confluent.*
2. *For all $\ell \longrightarrow r \in \mathcal{R}_{nc}$ and $\ell_c \longrightarrow r_c \in \mathcal{R}_c$, ℓ_c does not overlap $\hat{\ell}$.*
3. *If $\langle u_1, u_2, \simeq\rangle$ is an $\mathcal{R}_c$-cyclic critical pair of $\ell_1 \longrightarrow_{\alpha_1} r_1, \ell_2 \longrightarrow_{\alpha_2} r_2 \in \mathcal{R}_{nc}$, then*

$$u_1 \underset{\simeq;<\alpha_1}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_1' \xrightarrow[\alpha_2]{q_1}^{=} u_1'' \underset{\simeq;<\alpha_1,\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2'' \;{}^{=}\!\xleftarrow[\alpha_1]{q_1} u_2' \underset{\simeq;<\alpha_2}{\twoheadleftarrow\!\!\!\twoheadrightarrow} u_2$$

for some u_1', u_1'', u_2', u_2'' with q_i non-nested in u_i'.

Then $\mathcal{R}$ is confluent.

An example (not covered by any confluence tool of CoCo 2023)

Example

$0, \infty : \mathrm{tm}$

$\mathsf{S} : (\mathrm{tm})\mathrm{tm}$

$\mathrm{Eq}_3 : (\mathrm{tm}, \mathrm{tm}, \mathrm{tm})\mathrm{prop}$

$\top, \bot : \mathrm{prop}$

$1 : \mathrm{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{S}(\mathsf{Z})) \longrightarrow \mathrm{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{Z})$

$2 : \mathrm{Eq}_3(\mathsf{Y}, \mathsf{Y}, \mathsf{Y}) \longrightarrow \top$

$3 : \mathrm{Eq}_3(\mathsf{S}(\mathsf{V}), 0, \mathsf{V}) \longrightarrow \bot$

$4 : \infty \longrightarrow \mathsf{S}(\infty)$

An example (not covered by any confluence tool of CoCo 2023)

Example

$$\begin{array}{ll}
0, \infty : \mathrm{tm} & 1 : \mathsf{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{S}(\mathsf{Z})) \longrightarrow \mathsf{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{Z}) \\
\mathsf{S} : (\mathrm{tm})\mathrm{tm} & 2 : \mathsf{Eq}_3(\mathsf{Y}, \mathsf{Y}, \mathsf{Y}) \longrightarrow \top \\
\mathsf{Eq}_3 : (\mathrm{tm}, \mathrm{tm}, \mathrm{tm})\mathrm{prop} & 3 : \mathsf{Eq}_3(\mathsf{S}(\mathsf{V}), 0, \mathsf{V}) \longrightarrow \bot \\
\top, \bot : \mathrm{prop} & 4 : \infty \longrightarrow \mathsf{S}(\infty)
\end{array}$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$, hence $\mathcal{R}_c = \{\infty \longrightarrow \mathsf{S}(\infty)\}$.

An example (not covered by any confluence tool of CoCo 2023)

Example

$$
\begin{array}{ll}
0, \infty : \mathrm{tm} & 1 : \mathrm{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{S}(\mathsf{Z})) \longrightarrow \mathrm{Eq}_3(\mathsf{S}(\mathsf{X}), \mathsf{X}, \mathsf{Z}) \\
\mathsf{S} : (\mathrm{tm})\mathrm{tm} & 2 : \mathrm{Eq}_3(\mathsf{Y}, \mathsf{Y}, \mathsf{Y}) \longrightarrow \top \\
\mathrm{Eq}_3 : (\mathrm{tm}, \mathrm{tm}, \mathrm{tm})\mathrm{prop} & 3 : \mathrm{Eq}_3(\mathsf{S}(\mathsf{V}), 0, \mathsf{V}) \longrightarrow \bot \\
\top, \bot : \mathrm{prop} & 4 : \infty \longrightarrow \mathsf{S}(\infty)
\end{array}
$$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$, hence $\mathcal{R}_c = \{\infty \longrightarrow \mathsf{S}(\infty)\}$.

Confluence of $\mathcal{R}_c$ and absence of overlaps of $\infty \longrightarrow \mathsf{S}(\infty)$ into $\mathcal{R}_{nc}$ are easy to verify.

An example (not covered by any confluence tool of CoCo 2023)

Example

$0, \infty : \mathrm{tm}$ $\qquad$ $1 : \mathsf{Eq}_3(\mathsf{S}(\mathtt{X}), \mathtt{X}, \mathsf{S}(\mathtt{Z})) \longrightarrow \mathsf{Eq}_3(\mathsf{S}(\mathtt{X}), \mathtt{X}, \mathtt{Z})$

$\mathsf{S} : (\mathrm{tm})\mathrm{tm}$ $\qquad$ $2 : \mathsf{Eq}_3(\mathtt{Y}, \mathtt{Y}, \mathtt{Y}) \longrightarrow \top$

$\mathsf{Eq}_3 : (\mathrm{tm}, \mathrm{tm}, \mathrm{tm})\mathrm{prop}$ $\qquad$ $3 : \mathsf{Eq}_3(\mathsf{S}(\mathtt{V}), 0, \mathtt{V}) \longrightarrow \bot$

$\top, \bot : \mathrm{prop}$ $\qquad$ $4 : \infty \longrightarrow \mathsf{S}(\infty)$

We have $\mathcal{S}_c = \{\mathrm{tm}\}$, hence $\mathcal{R}_c = \{\infty \longrightarrow \mathsf{S}(\infty)\}$.

Confluence of $\mathcal{R}_c$ and absence of overlaps of $\infty \longrightarrow \mathsf{S}(\infty)$ into $\mathcal{R}_{nc}$ are easy to verify.

Only one cyclic critical pair, between 1 and 2 (ignoring symmetric counterparts):

$$\langle \mathsf{Eq}_3(\mathsf{S}(\mathtt{W}_1), \mathtt{W}_1, \mathtt{W}_3), \top, \simeq \rangle$$

with $\mathtt{W}_1 \simeq \mathtt{W}_3 \simeq \mathsf{S}(\mathtt{W}_3)$, which closes as $\mathsf{Eq}_3(\mathsf{S}(\mathtt{W}_1), \mathtt{W}_1, \mathtt{W}_3) \simeq \mathsf{Eq}_3(\mathtt{W}_1, \mathtt{W}_1, \mathtt{W}_1) \xrightarrow{\varepsilon} \top$.

The combination criterion

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \rightarrow r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \to r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be (non-necessarily disjoint) rewrite systems such that:

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \longrightarrow r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be (non-necessarily disjoint) rewrite systems such that:

1. *$\mathcal{R}_1$ and $\mathcal{R}_2$ are both confluent.*

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \to r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be (non-necessarily disjoint) rewrite systems such that:

1. *$\mathcal{R}_1$ and $\mathcal{R}_2$ are both confluent.*
2. *$\mathcal{R}_2|_{\mathcal{S}_{wc}(\mathcal{R}_1)} = \mathcal{R}_1|_{\mathcal{S}_{wc}(\mathcal{R}_2)} = \emptyset$.*

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \longrightarrow r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be (non-necessarily disjoint) rewrite systems such that:

1. *$\mathcal{R}_1$ and $\mathcal{R}_2$ are both confluent.*
2. $\mathcal{R}_2|_{\mathcal{S}_{wc}(\mathcal{R}_1)} = \mathcal{R}_1|_{\mathcal{S}_{wc}(\mathcal{R}_2)} = \emptyset$.
3. *There are no critical pairs between rules of $\mathcal{R}_1$ and $\mathcal{R}_2$.*

The combination criterion

We define the *weakly confined sorts* $\mathcal{S}_{wc}(\mathcal{R})$ by $s \in \mathcal{S}_{wc}(\mathcal{R})$ iff, for some $s' \succeq s$ and $\ell \to r \in \mathcal{R}$, some metavariable of sort s' occurs non-linearly in ℓ.

Theorem

Let $\mathcal{R}_1, \mathcal{R}_2$ be (non-necessarily disjoint) rewrite systems such that:

1. *$\mathcal{R}_1$ and $\mathcal{R}_2$ are both confluent.*
2. $\mathcal{R}_2|_{\mathcal{S}_{wc}(\mathcal{R}_1)} = \mathcal{R}_1|_{\mathcal{S}_{wc}(\mathcal{R}_2)} = \emptyset$.
3. *There are no critical pairs between rules of $\mathcal{R}_1$ and $\mathcal{R}_2$.*

Then $\mathcal{R}_1 \cup \mathcal{R}_2$ is confluent.

An example

Example

$$@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm}$$

$$\lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm}$$

$$\top : \mathrm{prop}$$

$$\mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop}$$

$$\neg : (\mathrm{prop})\mathrm{prop}$$

$$\wedge, \vee : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}$$

$$1 : @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\}$$

$$2 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$$

$$3 : \neg(\neg(\mathsf{A})) \longrightarrow \mathsf{A}$$

$$4 : \neg(\wedge(\mathsf{X}, \mathsf{Y})) \longrightarrow \vee(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$

$$5 : \neg(\vee(\mathsf{X}, \mathsf{Y})) \longrightarrow \wedge(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$

An example

Example

$@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm}$

$\lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm}$

$\top : \mathrm{prop}$

$\mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop}$

$\neg : (\mathrm{prop})\mathrm{prop}$

$\wedge, \vee : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}$

$1 : @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\}$

$2 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$

$3 : \neg(\neg(\mathsf{A})) \longrightarrow \mathsf{A}$

$4 : \neg(\wedge(\mathsf{X}, \mathsf{Y})) \longrightarrow \vee(\neg(\mathsf{X}), \neg(\mathsf{Y}))$

$5 : \neg(\vee(\mathsf{X}, \mathsf{Y})) \longrightarrow \wedge(\neg(\mathsf{X}), \neg(\mathsf{Y}))$

To prove confluence, let us take $\mathcal{R}_1$ with rules 1 and 2, and $\mathcal{R}_2$ with rules 3, 4 and 5.

An example

Example

$$@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm}$$

$$\lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm}$$

$$\top : \mathrm{prop}$$

$$\mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop}$$

$$\neg : (\mathrm{prop})\mathrm{prop}$$

$$\wedge, \vee : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}$$

$$1 : @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\}$$

$$2 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$$

$$3 : \neg(\neg(\mathsf{A})) \longrightarrow \mathsf{A}$$

$$4 : \neg(\wedge(\mathsf{X}, \mathsf{Y})) \longrightarrow \vee(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$

$$5 : \neg(\vee(\mathsf{X}, \mathsf{Y})) \longrightarrow \wedge(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$

To prove confluence, let us take $\mathcal{R}_1$ with rules 1 and 2, and $\mathcal{R}_2$ with rules 3, 4 and 5.

Confluence of $\mathcal{R}_1$ follows by second criterion, and of $\mathcal{R}_2$ by critical pair criterion.

An example

Example

$@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm}$

$\lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm}$

$\top : \mathrm{prop}$

$\mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop}$

$\neg : (\mathrm{prop})\mathrm{prop}$

$\wedge, \vee : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}$

$1 : @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\}$

$2 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$

$3 : \neg(\neg(\mathsf{A})) \longrightarrow \mathsf{A}$

$4 : \neg(\wedge(\mathsf{X}, \mathsf{Y})) \longrightarrow \vee(\neg(\mathsf{X}), \neg(\mathsf{Y}))$

$5 : \neg(\vee(\mathsf{X}, \mathsf{Y})) \longrightarrow \wedge(\neg(\mathsf{X}), \neg(\mathsf{Y}))$

To prove confluence, let us take $\mathcal{R}_1$ with rules 1 and 2, and $\mathcal{R}_2$ with rules 3, 4 and 5.

Confluence of $\mathcal{R}_1$ follows by second criterion, and of $\mathcal{R}_2$ by critical pair criterion.

We have $\mathcal{S}_{wc}(\mathcal{R}_1) = \{\mathrm{tm}\}$ and $\mathcal{S}_{wc}(\mathcal{R}_2) = \emptyset$, so $\mathcal{R}_2|_{\mathcal{S}_{wc}(\mathcal{R}_1)} = \mathcal{R}_1|_{\mathcal{S}_{wc}(\mathcal{R}_2)} = \emptyset$.

An example

Example

$$@ : (\mathrm{tm}, \mathrm{tm})\mathrm{tm}$$
$$\lambda : (\{\mathrm{tm}\}\mathrm{tm})\mathrm{tm}$$
$$\top : \mathrm{prop}$$
$$\mathrm{Eq} : (\mathrm{tm}, \mathrm{tm})\mathrm{prop}$$
$$\neg : (\mathrm{prop})\mathrm{prop}$$
$$\wedge, \vee : (\mathrm{prop}, \mathrm{prop})\mathrm{prop}$$

$$1 : @(\lambda(x.\mathsf{T}\{x\}), \mathsf{U}) \longrightarrow \mathsf{T}\{\mathsf{U}\}$$
$$2 : \mathrm{Eq}(\mathsf{N}, \mathsf{N}) \longrightarrow \top$$
$$3 : \neg(\neg(\mathsf{A})) \longrightarrow \mathsf{A}$$
$$4 : \neg(\wedge(\mathsf{X}, \mathsf{Y})) \longrightarrow \vee(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$
$$5 : \neg(\vee(\mathsf{X}, \mathsf{Y})) \longrightarrow \wedge(\neg(\mathsf{X}), \neg(\mathsf{Y}))$$

To prove confluence, let us take $\mathcal{R}_1$ with rules 1 and 2, and $\mathcal{R}_2$ with rules 3, 4 and 5.

Confluence of $\mathcal{R}_1$ follows by second criterion, and of $\mathcal{R}_2$ by critical pair criterion.

We have $\mathcal{S}_{wc}(\mathcal{R}_1) = \{\mathrm{tm}\}$ and $\mathcal{S}_{wc}(\mathcal{R}_2) = \emptyset$, so $\mathcal{R}_2|_{\mathcal{S}_{wc}(\mathcal{R}_1)} = \mathcal{R}_1|_{\mathcal{S}_{wc}(\mathcal{R}_2)} = \emptyset$.

No critical pairs between $\mathcal{R}_1$ and $\mathcal{R}_2$.

Conclusion

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams.*
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams.*
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

See the paper for:

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing $\underline{\text{cyclic}}$ critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

See the paper for:

- Proofs (theorems 1 and 2 follow from a final unifying theorem).

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

See the paper for:

- Proofs (theorems 1 and 2 follow from a final unifying theorem).
- First ideas for automating the criteria.

Conclusion

Three criteria for higher-order non-left-linear confluence, based on sort separation:

- **Theorem 1**: Supposing confluence and termination of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 2**: Supposing confluence ~~and termination~~ of $\mathcal{R}_c$, conclude confluence of $\mathcal{R}$ by closing cyclic critical pairs in $\mathcal{R}_{nc}$ with von Oostrom's *decreasing diagrams*.
- **Theorem 3**: A combination criterion for when confined sorts do not interact.

See the paper for:

- Proofs (theorems 1 and 2 follow from a final unifying theorem).
- First ideas for automating the criteria.

Thank you for your attention!