Impredicativity, Cumulativity and Product Covariance in the Logical Framework Dedukti

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FSCD 2024

July 11

Dedukti

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A framework for sharing proofs

- Fermat's Little Theorem in Matita $\rightsquigarrow$ in Coq, PVS, Lean (Thiré 2018)
- Euclids's first book in Coq $\rightsquigarrow$ in Matita, PVS, Lean (Géran 2021)
- Betrand's "Postulate" in Matita $\rightsquigarrow$ in Agda (Felicissimo *et al.* 2023)
- etc

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$$\Gamma, x : A \vdash_{\mathrm{CC}} P \ : \ \mathsf{Prop} \qquad \rightsquigarrow \qquad \Gamma \vdash_{\mathrm{CC}} \Pi x : A.P \ : \ \mathsf{Prop} \qquad \text{for all } A, \Gamma$$

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Notation From now on, I write U_0 for Prop and U_{n+1} for Type_n

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Soundness Fixes aforementioned problems by adopting different technique
Soundness is stated and proved in terms of inverse translation function

The Theory $\mathbb{T}_{\mathbf{CC}}$

The basis of the encoding: universes

$$\mathbb{N} : \mathbf{Type}$$
$$0 : \mathbb{N}$$
$$S : \mathbb{N} \to \mathbb{N}$$

$$U : (l : \mathbb{N}) \to \mathbf{Type}$$

Let $\underline{n} := S^n\ 0$. Then U_0 (Prop) represented by $U_{\underline{0}}$, and U_{n+1} (Type_n) represented by $U_{\underline{n+1}}$

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$$\mathfrak{A} : \mathbb{N} \to \mathbb{N}$$
$$\ldots$$
$$\mathrm{u} : (l : \mathbb{N}) \to \mathrm{U}_{(\mathfrak{A}\ l)}$$
$$\mathrm{El}_{(\_)}\ \mathrm{u}_1 \longmapsto \mathrm{U}_1$$

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$$\mathfrak{A} : \mathbb{N} \to \mathbb{N} \qquad \mathfrak{R} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\ldots \qquad \ldots$$
$$\mathrm{u} : (l : \mathbb{N}) \to \mathrm{U}_{(\mathfrak{A}\ l)} \qquad \pi : (l_1\ l_2 : \mathbb{N}) \to (A : \mathrm{U}_{l_1}) \to (B : \mathrm{El}_{l_1}\ A \to \mathrm{U}_{l_2}) \to \mathrm{U}_{(\mathfrak{R}\ l_1\ l_2)}$$
$$\mathrm{El}_{(\_)}\ \mathrm{u}_1 \longmapsto \mathrm{U}_1 \qquad \mathrm{El}_{(\_)}\ (\pi_{1_1,1_2}\ \mathsf{A}\ \lambda x.\mathsf{B}\{x\}) \longmapsto (x : \mathrm{El}_{1_1}\ \mathsf{A}) \to \mathrm{El}_{1_2}\ \mathsf{B}\{x\}$$

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Cumulativity subtype relation $\subset$ Defined by

1. "Simple" cumulativity: $\mathsf{U}_n \subset \mathsf{U}_m$ when $n \leq m$
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We can then define the lift as

$$\uparrow : (l : \mathbb{N}) \to (D : \mathsf{Tele}) \to (D \Rightarrow l) \to (D \Rightarrow (\mathsf{S}\ l))$$

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In a setting without product covariance, Assaf identified the following equations, where $\uparrow_{\underline{n}}^{m} D\ t$ is a notation for $\uparrow_{\underline{m-1}} D\ (\ldots(\uparrow_{\underline{n}} D\ t)\ldots)$ when $n \leq m$

$$\pi_{\underline{1+n,m}}\ (\uparrow_{\underline{n}} \diamond A)\ (\lambda x.B) \equiv \uparrow_{\underline{\mathfrak{R}(n,m)}}^{\mathfrak{R}(1+n,m)} \diamond\ (\pi_{\underline{n,m}}\ A\ (\lambda x.B))$$

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Problematic as (finite) rewrite rules Multi-step lift $\uparrow_{\underline{n}}^{\underline{m}}$ only a notation

How to know correct number of lifts when applying the rule?

A factorizing π

Solution We replace the constant π with the following one

$$\pi : (l_0\ l_1\ l_2 : \mathbb{N}) \to (A : \mathrm{U}_{(l_0 + l_1)}) \to (B : \mathrm{El}_{(l_0 + l_1)}\ A \to \mathrm{U}_{(l_0 + l_2)}) \to \mathrm{U}_{(\mathfrak{R}\ (l_0 + l_1)\ (l_0 + l_2))}$$

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If A in $\mathrm{U}_{\underline{n_A}}$ and B in $\mathrm{U}_{\underline{n_B}}$, common part of n_A and n_B can be *factorized* with rule:

$$\pi^{l_0}_{(\mathrm{S}\ l_1),(\mathrm{S}\ l_2)}\ \mathsf{A}\ \mathsf{B} \longmapsto \pi^{(\mathrm{S}\ l_0)}_{l_1,l_2}\ \mathsf{A}\ \mathsf{B}$$

Now, if $\pi^{n_0}_{\underline{n_1},\underline{n_2}}\ A\ \lambda x.B$ in normal form, then $n_1 = 0$ or $n_2 = 0$

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We can consider all possible cases, in each one we know how many lifts to insert

$$\pi^{0}_{(\mathrm{S}\ 1),0}\ (\uparrow_{\_} \diamond \mathsf{A})\ \mathsf{B} \longmapsto \pi^{0}_{1,0}\ \mathsf{A}\ \mathsf{B} \qquad (\ldots)$$

$$\pi^{l_1}_{0,(\mathrm{S}\ l_2)}\ \mathsf{A}\ (\lambda x.\uparrow_{\_} \diamond \mathsf{B}\{x\}) \longmapsto \uparrow_{(l_1 + l_2)} \diamond (\pi^{l_1}_{0,l_2}\ \mathsf{A}\ (\lambda x.\mathsf{B}\{x\})) \qquad (\ldots)$$

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Finally, codes in different universes may represent the same type, so we add:

$$\mathrm{El}_1\ (\uparrow_\_ \diamond \mathtt{A}) \longmapsto \mathrm{El}_{(\mathrm{P}\ 1)}\ \mathtt{A}$$
$$(\uparrow_\_ \diamond \mathtt{A})\ _1 \triangleleft \mathtt{D} \longmapsto \mathtt{A}\ _{(\mathrm{P}\ 1)} \triangleleft \mathtt{D}$$

Confluence and subject reduction

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$$\beta\mathcal{R}_{\text{cc}} \quad = \quad \beta\mathcal{R}_1 \quad \cup \quad \mathcal{R}_2$$

No critical pairs between $\beta\mathcal{R}_1$ and $\mathcal{R}_2$, and both are left-linear

By Van Oostrom and Van Raamsdonk's *orthogonal combination criterion*,
it suffices to show that both $\beta\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent

Confluence of $\mathcal{R}_2$: the easy part

Critical pairs of $\mathcal{R}_2$ all close, verified with SOL and CSI$^{\text{ho}}$

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By Newman's Lemma, $\mathcal{R}_2$ is confluent

Confluence of $\beta\mathcal{R}_1$: the hard part

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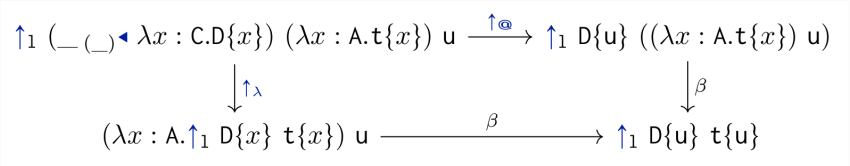

1

Confluence of $\beta\mathcal{R}_1$: the hard part

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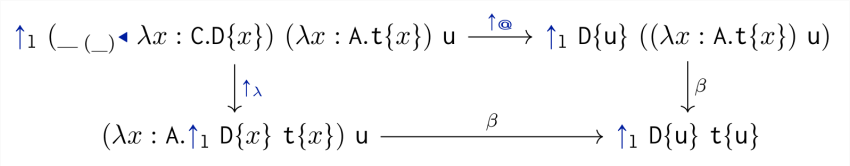

Solution We can still show that *orthogonal rewriting*[1] $\Longrightarrow_{\beta\mathcal{R}_1}$ satisfies diamond prop.

By the usual argument, $\Longrightarrow^*_{\beta\mathcal{R}_1} = \longrightarrow^*_{\beta\mathcal{R}_1}$, hence $\beta\mathcal{R}_1$ is confluent

[1] Also known as *developments* or *multi-step rewriting* or *simultaneous rewriting*

Subject reduction

Problem Not all rules satisfy subject reduction unconditionally

$$\pi^{(\mathsf{S}\ 1_1)}_{0,1_2}\ (\uparrow_{\_} \diamond \mathsf{A})\ \mathsf{B} \longmapsto \pi^{1_1}_{0,(\mathsf{S}\ 1_2)}\ \mathsf{A}\ \mathsf{B}$$

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Only preserves typing if $\mathsf{S}\ (\mathtt{l}_1 \vee (\mathtt{l}_1 + \mathtt{l}_2)) \equiv \mathtt{l}_1 \vee \mathsf{S}\ (\mathtt{l}_1 + \mathtt{l}_2)$

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Solution We can still show subject reduction for the *guarded terms* where each occurrence of π is of the form $\pi^{n_0}_{\underline{n_1},\underline{n_2}}$ for $n_0, n_1, n_2 \in \mathbb{N}$

Such terms are the only ones we need in the encoding

Soundness and Conservativity

The translation function

Problem Defining translation function $[\![-]\!] : \mathrm{CC} \to \textsc{Dedukti}$ is hard

Cumulativity is implicit in CC, so $[\![-]\!]$ needs to figure out where to insert lifts $\uparrow$

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Solution We state soundness using a (partial) *inverse translation function*:

$$|-| : \textsc{Dedukti} \dashrightarrow \mathrm{CC}$$

$$|x| := x \qquad |\pi^{0}_{\underline{n},\underline{m}}\ A\ (\lambda x.B)| := \Pi x : |A|.|B|$$

$$|\mathrm{u}_{\underline{n}}| := \mathrm{U}_n \qquad |\uparrow_{\underline{n}}\ D\ t| := |t|$$

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Define the *object terms* as the $\textsc{Dedukti}$ terms t for which $|t|$ is defined

Soundness

The main lemma:

Coherence For t_1, t_2 object terms with $\Gamma \vdash t_i : A$, if $|t_1| = |t_2|$ then $t_1 \equiv t_2$

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The direct translation function can then be extract from the proof

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Using inverse translation function, conservativity becomes

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In practical applications, only object terms are used anyway

It should be possible to prove full conservativity (see concl. of the paper)

Conclusion

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An encoding of CC satisfying

	This work
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Confluence	✓
Conservativity	✓⋆

⋆: Only for object terms.

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Challenge for confluence checkers Proving confluence of $\beta\mathcal{R}_{\text{CC}}$ automatically

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Thank you!