Consolidating Equality in a Proof Irrelevant Universe

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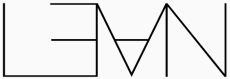

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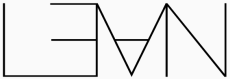

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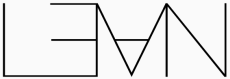

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$$\frac{A : \mathsf{Type} \qquad P : \mathsf{Prop}}{\forall (x : A).P : \mathsf{Prop}} \qquad\qquad \frac{P : \mathsf{Prop} \qquad p : P \qquad q : P}{p \equiv q : P}$$

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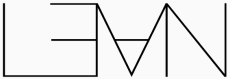

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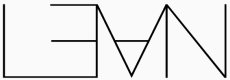

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Yet, can be eliminated into *relevant* types: from $e : n =_{\mathsf{Nat}} m$ and $l : \mathsf{Vec}_A\ n$ we get $l' : \mathsf{Vec}_A\ m$

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Our goal Clarify the status of Lean's proof-irrelevant equality, used in practice

Problem 1: Decidability of conversion

Lean's equality can be eliminated into relevant types using *transport*:

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But proofs are irrelevant anyway!

Open problem [Abel & Coquand 2020] Is conversion still decidable?

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Natural questions Weaker form of canonicity, e.g. up to propositional equality?
Can the mechanism underlying `native_decide` be justified?

Towards a stronger computation rule?

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Question What about the previous questions in this stronger setting?

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Rest of the talk We focus only on the theory with TranspMotive, henceforth called $\mathsf{TT}_{\mathsf{sEq}}$

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Therefore, the fundamental theorem follows from:

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Problem Ruling out head mismatch cases (e.g., $\mathsf{Nat} \equiv \mathsf{Bool}$) needs *definitional inversion*



Decidability of conversion: the challenge

Usual proof strategy Define logical relation predicate $\Gamma \Vdash t : A$, ensuring t has normal form

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We extend the theory with a trivial conversion operator:

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Observation For judgements not mentioning conv, extension does not change derivability

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But computationally, treat conv like cast of *observational equality*:

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Indeed, head mismatch cases don't need definitional inversion:
if $\mathsf{Nat} \equiv \mathsf{Bool}$ and $t : \mathsf{Nat}$ then $\mathsf{conv}_{\mathsf{Nat},\mathsf{Bool}}\ t : \mathsf{Bool}$ is stuck, hence in the logical relation!

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LR Conv' $\Longrightarrow$ Fund Thm

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Canonicity If $\vdash t : \mathsf{Nat}$ then $\vdash t \equiv \mathsf{S}^k(\mathsf{0}) : \mathsf{Nat}$ for some k

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Too strong in practice: fails with axioms...

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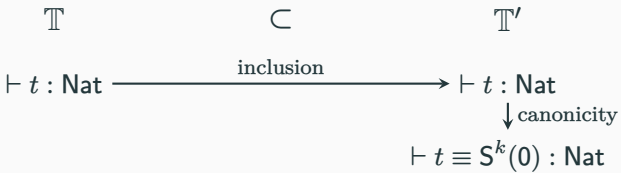

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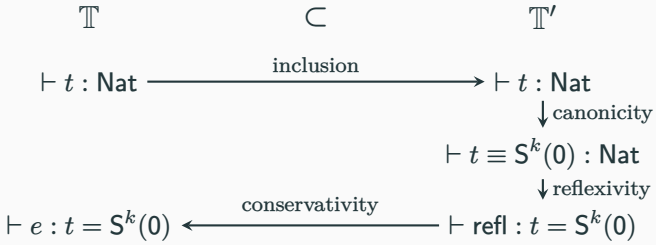

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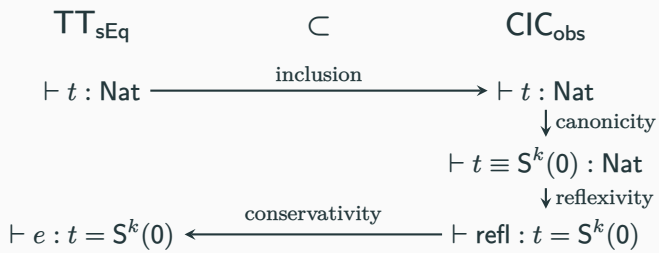

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Supports commonly used axioms: FunExt, $\mathsf{PropExt}$, LEM, ...

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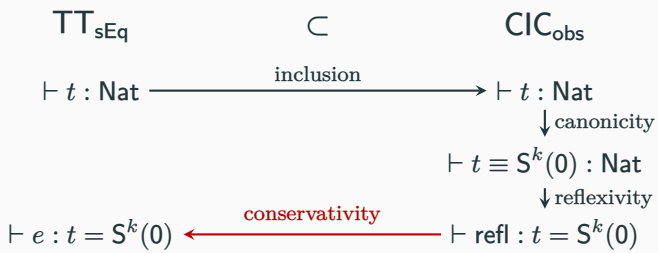

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In $\mathsf{CIC}_{\mathsf{obs}}$, equality is eliminated with cast, which computes by case analysis on types:

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where inj_1^{Π}, inj_2^{Π} are *injectivity axioms*, needed for cast to compute:

$$\mathsf{inj}_1^{\Pi} : (e : \Pi(x : A_1).B_1 = \Pi(x : A_2).B_2) \to A_2 = A_1$$
$$\mathsf{inj}_2^{\Pi} : (e : \Pi(x : A_1).B_1 = \Pi(x : A_2).B_2) \to (x_2 : A_2) \to$$
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Erasure $|-|$ into an untyped λ-calculus, removing proofs, types and transports:

$$e, e' ::= x \mid \lambda x.\, e \mid e_1\ e_2 \mid 0 \mid \mathsf{S}(e) \mid \mathsf{rec}_{\mathsf{Nat}}(e_1, x\, y.\, e_2, e_3) \mid \square$$

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Proof Refined propositional canonicity statement also gives *evaluation*: $t \Downarrow k$ in $\mathsf{CIC}_{\mathsf{obs}}$

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with call-by-name reduction $\mapsto$, and $e \Downarrow k$ if e evaluates to $\mathsf{S}^k(0)$

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for some e and (unique) k, under inherited technical conditions on the axioms

Proof Refined propositional canonicity statement also gives *evaluation*: $t \Downarrow k$ in $\mathsf{CIC}_{\mathsf{obs}}$

Extend $|-|$ with $|\mathsf{cast}\ A\ B\ e\ t| := |t|$, then show $\mathsf{CIC}_{\mathsf{obs}}$ step $t \longrightarrow u$ gives $|t| \mapsto |u|$ or $|t| = |u|$

Evaluation-via-erasure

Using prop. canonicity, we justify the transport erasure performed by `native_decide`

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By iterating over the reduction steps of $t \Downarrow k$ in $\mathsf{CIC}_{\mathsf{obs}}$, we get $|t| \Downarrow k$ in the λ-calculus

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Thank you for your attention!